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A Chord Approach for an Alternative Ruler and Compasses Construction of the 17-Side Regular Polygon

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Abstract. A new approach for stating and solving the problem of ruler and compasses construction of the regular polygon of 17 sides is presented. The approach is based on some relationships among certain functions of the lengths of the chords of the polygon. These relationships are used to derive the equations and to find their roots using only rational operations and the extraction of real square roots. The obtained expression allows a direct ruler and compasses construction of the polygon.

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If a chord of the unit circle subtends an angle 2θ at the centre of the circle, then the length of the chord is $2 \sin \theta$ and its distance from the centre of the circle is $\cos \theta$. Closely related to those geometrically significant quantities is the algebraically convenient quantity

$$y(\theta) = 4\cos^2\theta. \tag{1}$$

Under angle-doubling this quantity satisfies the equation:

$$y(2\theta) = 4\cos^2 2\theta = 4(2\cos^2\theta - 1)^2 = (4\cos^2\theta - 2)^2 = (y(\theta) - 2)^2. \tag{2}$$

Since $y(\psi) = y(\theta)$ if and only if $\psi \equiv \pm\theta (\text{mod } \pi)$, the six cycles

$$\begin{aligned}
&0 \Rightarrow 0\\
&\pi/3 \Rightarrow 2\pi/3 \sim \pi/3\\
&\pi/5 \Rightarrow 2\pi/5 \Rightarrow 4\pi/5 \sim \pi/5\\
&\pi/15 \Rightarrow 2\pi/15 \Rightarrow 4\pi/15 \Rightarrow 8\pi/15 \Rightarrow 16\pi/15 \sim \pi/15\\
&\pi/17 \Rightarrow 2\pi/17 \Rightarrow 4\pi/17 \Rightarrow 8\pi/17 \Rightarrow 16\pi/17 \sim \pi/17\\
&3\pi/17 \Rightarrow 6\pi/17 \Rightarrow 12\pi/17 \Rightarrow 24\pi/17 \Rightarrow 48\pi/17 \sim 3\pi/17
\end{aligned}$$

account for all $16 = 1 + 1 + 2 + 4 + 4 + 4$ roots of the equation

$$((((y-2)^2 - 2)^2 - 2)^2 - 2)^2 - y = 0 \tag{3}$$

which, after factoring, is transformed to

$$(y-4)(y-1)(y^2-3y+1)(y^4-9y^3+26y^2-24y+1)(y^8-15y^7 \\ +91y^6-286y^5+495y^4-462y^3+210y^2-36y+1)=0.$$

The quantities $y(\theta)$ for $\theta = 0,\ \pi/3,\ \pi/5,\ 2\pi/5,\ \pi/15,\ 2\pi/15,\ 4\pi/15,\ 8\pi/15$ are relatively accessible. They are the roots of the first four polynomials in the previous equation. Accordingly, the roots of (3) which are directly related to the 17-gon must satisfy

$$y^8-15y^7+91y^6-286y^5+495y^4-462y^3+210y^2-36y+1=0. \quad (4)$$

If we write the left side of (4) as the product of two fourth-degree polynomials with roots corresponding to the cycles that begin with $\pi/17$ and with $3\pi/17$, make use of the Vieta equations that relate the coefficients of the polynomial with its roots, and take into account property (2) as well as the identity

$$y(3\theta)=y(\theta)y(2\theta)-3y(\theta)-2y(2\theta)+8 \quad (5)$$

then, linking those two sets of roots, we find that the quantities $y(\theta)$ for $\theta = \pi/17,\ 2\pi/17,\ 4\pi/17,\ 8\pi/17$ satisfy the equation

$$y^4-\frac{15+\sqrt{17}}{2}y^3+\frac{39+5\sqrt{17}}{2}y^2-(18+3\sqrt{17})y+1=0 \quad (6)$$

while those belonging to the other cycle satisfy the conjugate equation

$$y^4-\frac{15-\sqrt{17}}{2}y^3+\frac{39-5\sqrt{17}}{2}y^2-(18-3\sqrt{17})y+1=0. \quad (7)$$

These equations lead to

$$y\left(\frac{\pi}{17}\right)=\frac{1}{8}\left[15+\sqrt{17}+\sqrt{34-2\sqrt{17}} \right. \\ \left. +2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}\right] \quad (8)$$

and hence

$$t=2\cos\frac{2\pi}{17}=y\left(\frac{\pi}{17}\right)-2 \\ =\frac{1}{8}\left[\sqrt{17}-1+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}\right]. \quad (9)$$

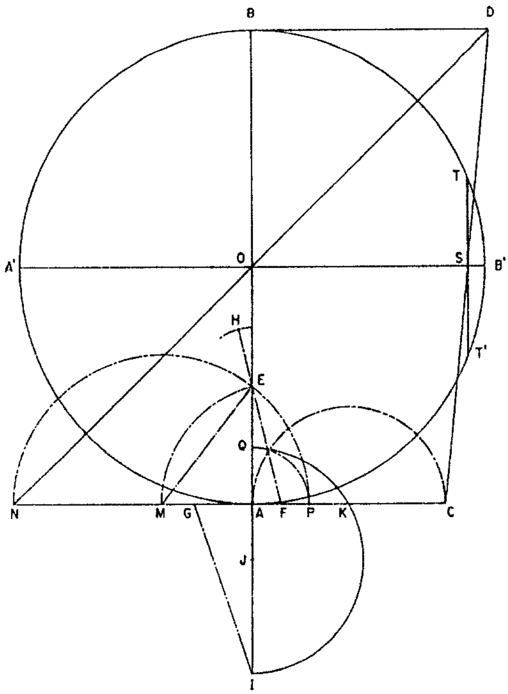

Fig. 1. Ruler and compasses construction of the 17-gon.

This quantity, represented by twice the length OS in Figure 1, can be used as the basis to construct the 17-gon.

In his original study of the 17-gon, Gauss found ([2], [4], [5]) that

$$t = \tfrac{1}{8}\Bigg[\sqrt{17} - 1 + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - 4\sqrt{34 + 2\sqrt{17}} - \tfrac{1}{2}(1 - \sqrt{17})\sqrt{34 - 2\sqrt{17}}}\Bigg]. \quad (10)$$

It is not hard to verify directly that

$$17 + 3\sqrt{17} = \sqrt{34 - 2\sqrt{17}}\,\sqrt{17 + 4\sqrt{17}} \quad (11)$$

and also that

$$4\sqrt{34 + 2\sqrt{17}} + \tfrac{1}{2}(1 - \sqrt{17})\,\sqrt{34 - 2\sqrt{17}} = \sqrt{170 + 38\sqrt{17}} = (-1 + \sqrt{17})\sqrt{17 + 4\sqrt{17}}. \quad (12)$$

Thus the two expressions for t are equal and suggest the third expression

$$t = \frac{1}{8}\left[\sqrt{34 - 2\sqrt{17}} + \sqrt{17} - 1 + 2\sqrt{\left(\sqrt{34 - 2\sqrt{17}} - \sqrt{17} + 1\right)\sqrt{17 + 4\sqrt{17}}}\right] \tag{13}$$

which can be taken as the basis of a convenient direct construction of the regular 17-gon.

Several constructions of the 17-gon have been given. One of the first was given by Richmond [6]. Klein [5] presents a method by von Standt and Schröter that uses the straight edge and one fixed circle. Dickson [2], Hall [3] and Smith [7] also present or quote constructions that use several circles and straight lines. Archibald, in his notes to Klein's book, as well as in a previous paper [1], also mentions several different constructions, some of which were reported by Gauss in 1825. Most of these constructions are based on the geometrical solution of the four quadratic equations obtained by Gauss. On the other hand, the direct construction of expression t was avoided, probably because of the complexity of the radicand in the big square root (see equation (10)). However, expression (13) allows a direct construction, as is explained in the following.

In a circle of radius unity (see Figure 1), construct two perpendicular diameters AB and $A'B'$ and at A and B draw tangents. For convenience, in the following we will express all lengths as multiples of one-eighth, so that radius has 8 units of length. Determine point E at the middle of AO, point G two units to the left of A, and point F one unit to the right of A. Observe that:

$$FE = \sqrt{17}.$$

Let the circle with centre F and radius FE cut the lower tangent at M. Observe that:

$$MA = \sqrt{17} - 1.$$

Observe also that:

$$ME = \sqrt{MA^2 + AE^2} = \sqrt{34 - 2\sqrt{17}}.$$

Now, let the circle with centre M and radius ME cut the lower tangent at N and at P. Observe that:

$$NA = \sqrt{34 - 2\sqrt{17}} + \sqrt{17} - 1$$

and

$$AP = \sqrt{34 - 2\sqrt{17}} - \sqrt{17} + 1.$$

Make $AQ = AP$. Now over the continuation of line FE make $EH = 2$, so that $FH = \sqrt{17} + 2$. Now let the circle with centre G and radius FH cut the prolongation of diameter AB at I. We see that

$$AI = \sqrt{GI^2 - AG^2}$$

but $GI = FH = \sqrt{17} + 2$ and $AG = 2$, so we obtain

$$AI = \sqrt{17 + 4\sqrt{17}}.$$

Determine point J at the middle of QI. Now let the circle with centre J and radius $(AQ + AI)/2$ cut the lower tangent at point K. We know that

$$KA^2 = AQ \cdot AI = \left(\sqrt{34 - 2\sqrt{17}} - \sqrt{17} + 1\right)\sqrt{17 + 4\sqrt{17}}.$$

Finally make $KC = KA$. Obviously we have

$$NC = t.$$

Line NO will cut the upper tangent at D. Then we join D with C. Line DC cuts OB' at S, for which $OS = t/2$ and the perpendicular at S will cut the circle in the points T and T'. $B'T$ and $B'T'$ are sides of the inscribed regular 17-gon.

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