# PROBLEMS AND SOLUTIONS.

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## A CONSTRUCTION OF THE REGULAR POLYGON OF SEVENTEEN SIDES.[1]

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Many methods have been devised for constructing the regular 17-gon with ruler and compasses. These are based on the fact, as demonstrated by Gauss, that the solution of the equation, $x^{17} - 1 = 0$, can be reduced to the solution of quadratics. This reduction is clearly discussed in the little book, Klein's *Famous Problems in Elementary Geometry*.[2] After removing the factor $(x - 1)$ the equation is reduced to the four following quadratics: $z_1 = 2\cos 2\pi/17$ is a root of

(1) $z^2 - y_1 z + y_4 = 0$ where $z_1 > z_2$.

Here $y_1$ is the larger root of the equation

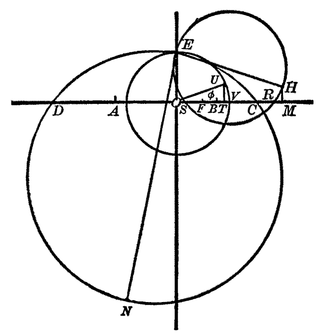
(2) $y^2 - x_1 y - 1 = 0$

and $y_4$ is the larger root of the equation

(3) $y^2 - x_2 y - 1 = 0;$

while $x_1$ and $x_2$ are the roots of the equation

(4) $x^2 + x - 4 = 0$ where $x_1 > x_2$.




We now obtain the required root $z_1$ by constructing the roots of these various equations. As a basis we use the method of constructing a quadratic's roots where, considering the equation of the form

$$x^2 - ax + b = 0,$$

a circle is erected on the line joining the points (0, 1) and $(a, b)$ as a diameter and the roots are the abscissas of the two intersections of the circle and the $X$ axis.

In the figure, $E$ is the point (0, 1) and $N$ is the point $(-1, -4)$ and thus the circle on the line $EN$ as a diameter solves the equation (4) and gives us

$$OC = x_1 \quad \text{and} \quad OD = x_2.$$

$A$ and $B$ are the mid-points of $OD$ and $OC$ respectively. $F$ is the intersection of the circle, whose center is $A$ and radius $AE$, with the $X$ axis, and thus gives us the larger root of equation (3)

$$OF = y_4.$$

[1] This article and the next one should be considered with problem 2745 below.

[2] Translated into English by Beman and Smith, Boston, 1897, pp. 24–32.

The reason for this is that the circle ordinarily used to construct these roots is symmetrical with the $X$ axis and thus we find the center by the simple bisection and do not have to draw this circle to find the one root that we need from this equation. In exactly the same manner $M$ is the intersection of the $X$-axis and the circle with center $B$ and radius $BE$. Hence we have the larger root of the equation (2),

$$OM = y_1.$$

Now erect a perpendicular to the $X$ axis at $M$ and lay off

$$MH = OF.$$

Thus the coördinates of the point $H$ are $(y_1, y_4)$. It is evident that the circle on $EH$ as a diameter intersects the $X$ axis so that $OS$ and $OR$ are the roots of equation (1) and since $OR$ is the larger,

$$OR = 2 \cos \frac{2\pi}{17}.$$

Having twice the cosine of the angle it is an easy matter to find the required angle. Bisect $OR$ in $T$ and draw the unit circle $VUE$. Construct the angle, $\varphi = 2\pi/17$ from the known cosine. Of course $UV$ subtending the angle $\varphi$ is the required side of the regular 17-gon.

## Gauss and the Regular Polygon of Seventeen Sides.

By R. C. Archibald, Brown University.

Recent publications call attention to new material[1] in connection with the history of the construction of the regular polygon of seventeen sides. The discovery that this construction could be effected with ruler and compasses only, was one of which Gauss was vastly proud[2] throughout his life and also, according to Sartorious von Waltershausen,[3] the one which decided him to dedicate his life to the study of mathematics. As it is recorded of Archimedes that he desired a sphere inscribed in a cylinder to be engraved on his tombstone, and similarly with Ludolph van Ceulen as to the value of $\pi$ to 35 places of decimals, and with

[1] *C. F. Gauss als Geometer* von P. Stäckel (*Materialien für eine wissenschaftliche Biographie von Gauss*, Heft V), Leipzig, 1918 (see pp. 78, 96); *Carl Friedrich Gauss Werke*, Band 10, Abteilung 1, Leipzig, 1917 (see pp. 3–4, 120–126, 487 and the facsimile of Gauss's notebook 1796–1814).

[2] The very first entry in his notebook "1796 Mart. 30–1814 Jul. 9" is: "1796. Principia quibus innitur sectio circuli ac divisibilitas eiusdem geometrica in septemdecim partes etc. Mart. 30. Brunsv[igae]." Again, in his own copy of his *Disquisitiones Arithmeticae* he wrote the following note in the margin beside article 365: "Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30." And finally, on page 77 of "Scheda Af," begun in 1801, Gauss brought together a number of dates which were of importance to him. The first four of these were: 1. Jan., 1801, Ceres discovered; 28. March, 1802 Pallas discovered; 19. Feb., 1803 Pallas rediscovered; 30. March, 1796, "Construction des 17 Ecks."

[3] *Gauss zum Gedächtnis*, Leipzig, 1856, p. 16: "Diese Entdeckung, welche Gauss bis zum Ende seines Lebens sehr hoch schätzte, ist es vornehmlich gewesen, welche seinem Leben eine bestimmte Richtung gegeben hat, denn von jenem Tage an war er fest entschlossen, nur der Mathematik sein Leben zu widmen."