# To construct a regular polygon of 17 sides.

By

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A considerable simplification can be effected in the geometrical constructions commonly quoted if it is realized that the two first steps of the solution of $x^{17}=1$ amount to no more than the division of a certain angle into four equal parts. The following method was indicated briefly in the Quarterly Journal of Mathematics in 1892. For the solution of $x^{17}=1$ and an account of the problem it is sufficient to refer to Klein's *Vorträge über ausgewählte Fragen der Elementargeometrie* and its translations, or to Enriques' *Fragen der Elementargeometrie* [Deutsche Ausgabe von H. Fleischer], II, p. 175—188.

Let $\alpha$ be the angle such that $17\alpha = 2\pi$, and let

$$\varepsilon = \cos\alpha + i\sin\alpha;$$

the imaginary roots of $x^{17}=1$ are the first sixteen powers of $\varepsilon$, which for the purpose of solution are arranged in a cycle such that each root is the cube of the preceding.

The first step in the solution is to evaluate

$$\lambda_1 = \Sigma\varepsilon^n \text{ when } n = 3,\ 10,\ 5,\ 11,\ 14,\ 7,\ 12,\ 6;$$

$$\lambda_2 = \Sigma\varepsilon^n \text{ when } n = 9,\ 13,\ 15,\ 16,\ 8,\ 4,\ 2,\ 1:$$

it is shewn that $\lambda_1$ and $\lambda_2$ are roots of the quadratic

(1) $$z^2 + z - 4 = 0.$$

The second step is to determine

$$\mu_1 = \Sigma\varepsilon^n \text{ when } n = 3,\ 5,\ 14,\ 12, \quad = 2\cos 3\alpha + 2\cos 5\alpha;$$

$$\mu_2 = \Sigma\varepsilon^n \text{ when } n = 9,\ 15,\ 8,\ 2, \quad = 2\cos 2\alpha + 2\cos 8\alpha;$$

$$\mu_3 = \Sigma\varepsilon^n \text{ when } n = 10,\ 11,\ 7,\ 6, \quad = 2\cos 6\alpha + 2\cos 7\alpha;$$

$$\mu_4 = \Sigma\varepsilon^n \text{ when } n = 13,\ 16,\ 4,\ 1, \quad = 2\cos\alpha + 2\cos 4\alpha:$$

it is shewn that $\mu_1$ and $\mu_3$ are roots of the quadratic

(2) $$z^2 - \lambda_1 z - 1 = 0,$$

and that $\mu_2$ and $\mu_4$ are roots of

(3) $$z^2 - \lambda_2 z - 1 = 0.$$

Now if $4D$ denote the acute angle whose tangent is 4, equation (1) may be written

$$z^2 + 4z \cot 4D - 4 = 0,$$

and its roots $\lambda_1$ and $\lambda_2$ must be $-2\cot 2D$ and $2\tan 2D$. Similarly equations (2) and (3) may be written

$$z^2 + 2z\cot 2D - 1 = 0, \quad \text{and} \quad z^2 - 2z\tan 2D - 1 = 0;$$

the values of $\mu_1, \mu_2, \mu_3, \mu_4$ must be

$$\tan D, \quad \tan\left(D + \tfrac{1}{4}\pi\right), \quad \tan\left(D + \tfrac{1}{2}\pi\right) \text{ and } \tan\left(D + \tfrac{3}{4}\pi\right),$$

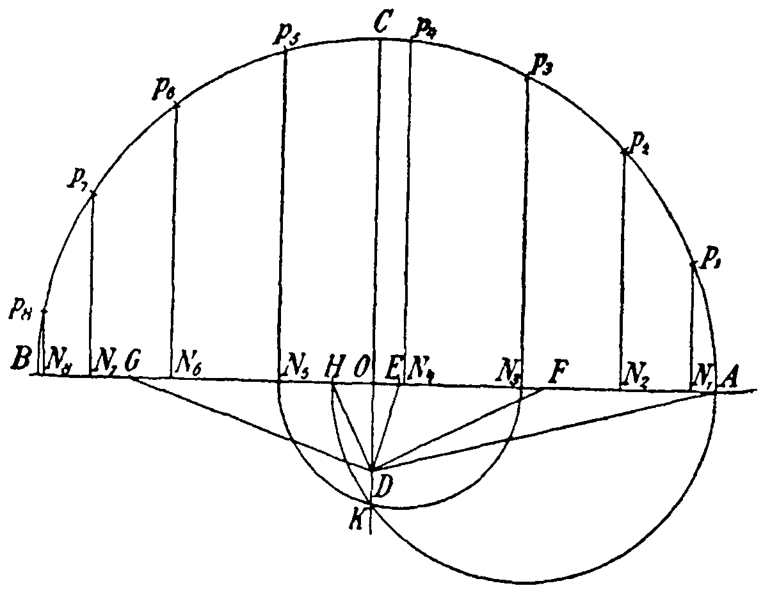
and a very rough numerical calculation serves to shew which tangent represents each of the quantities $\mu$. Finally we have the following results:

$$2\cos 3\alpha + 2\cos 5\alpha = 2\cos\alpha \cdot 2\cos 4\alpha = \tan D;$$
$$2\cos\alpha + 2\cos 4\alpha = 2\cos 6\alpha \cdot 2\cos 7\alpha = \tan\left(D + \tfrac{1}{4}\pi\right);$$
$$2\cos 6\alpha + 2\cos 7\alpha = 2\cos 2\alpha \cdot 2\cos 8\alpha = \tan\left(D + \tfrac{1}{2}\pi\right);$$
$$2\cos 2\alpha + 2\cos 8\alpha = 2\cos 3\alpha \cdot 2\cos 5\alpha = \tan\left(D + \tfrac{3}{4}\pi\right),$$

and from them we derive a method for the construction of a regular polygon of 17 sides.

## Construction.

Let $O$ be the centre and $AOB$ a diameter of a circle.




(1) Draw $COD$ at right angles to $AB$ and make $OD = \frac{1}{4} OA$.

(2) Join $AD$ and make the angle $ODE = \frac{1}{4}$ of the angle $ODA$.

(3) Draw lines through $D$ at angles of half a right angle starting from $DE$, and let these lines meet $AB$ in $E, F, G, H$.

Then, if $P_1, P_2, \cdots, P_8$ are eight vertices of the polygon reckoned in order from the vertex $A$, and $N_1, N_2, \cdots, N_8$ their projections on $AB$,

$$ON_1 + ON_4 = 2 \cdot OF; \quad ON_1 \cdot ON_4 = OA \cdot OE;$$
$$ON_6 + ON_7 = 2 \cdot OG; \quad ON_6 \cdot ON_7 = OA \cdot OF;$$
$$ON_2 + ON_8 = 2 \cdot OH; \quad ON_2 \cdot ON_8 = OA \cdot OG;$$
$$ON_3 + ON_5 = 2 \cdot OE; \quad ON_3 \cdot ON_5 = OA \cdot OH.$$

From these results it is easy to devise a means of determining each of the points $N_1, N_2, \cdots, N_8$. Thus, if the circle drawn upon $AH$ as diameter cuts $CD$ in $K$, a circle with centre $E$ and radius $EK$ will meet $AB$ in $N_3$ and $N_5$.

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When $n$ is a prime of the form $4\alpha + 1$, the quadratic corresponding to the subdivision of the $4\alpha$ roots of $x^n = 1$ into two sets may be solved graphically by bisecting an angle whose tangent is $(n-1)^{\frac{1}{2}}$. Successive bisections of this angle enable us to solve graphically the quadratics which give subdivisions of the roots into 4, 8, 16, $\cdots$ sets so long as (a) the number of roots in a set is even, and (b) the quadratics have their absolute terms rational: but the latter condition is seldom satisfied. Even at the second step (that of subdivision into four sets), the known forms of the quadratics prove that (b) cannot be satisfied when $n-1$ is not a perfect square; and in the case of $n = 257$, the process will be found to fail at this stage. Thus, although the foregoing construction was not obtained at haphazard, other cases to which a similar method can be applied are disappointingly rare.

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