INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 12 (2023), No. 4, 28 - 38

NEW INEQUALITIES IN A BICENTRIC QUADRILATERAL

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Abstract. In this paper we will demonstrate some new inequalities in bicentric quadrilateral. For two inequalities we will also give a geometric interpretation.

1. Introduction

In this section we will recall some known results, which we will use in the following.
In a bicentric quadrilateral we denote the lenghts of the sides with a, b, c, d. F the area, r the radius of the inscribed circle, respectively R the radius of circumscribed circle and with $s = \frac{a+b+c+d}{2}$ the semiperimeter.

Theorem 1 (see [4] pages 39-54, or [7] page 164)**.** *In a bicentric quadrilateral are true the following equalities*

$$\sigma_1 = \sum a = 2s \tag{1}$$

$$\sigma_2 = \sum ab = s^2 + 2r^2 + 2r\sqrt{4R^2+r^2} \tag{2}$$

$$\sigma_3 = \sum abc = s\left(2r^2 + 2r\sqrt{4R^2+r^2}\right) \tag{3}$$

and

$$\sigma_4 = abcd = F^2 = s^2r^2 \tag{4}$$

We denote with $s_1 = \sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$ and $s_2 = \sqrt{4R^2+r^2}+r$.

Keywords and phrases: Ineq. in a bicentric quadrilateral.
(2021) Mathematics Subject Classification: 26D15; 26D20
Received: 03.05.2023. In revised form: 26.09.2023. Accepted: 05.06.2023

Theorem 2 (Blundon-Eddy Inequality, see [3], or [4] page 58, or [7] pages 168-171)**.** *A bicentric quadrilateral is given. The inequality*

$$s_1 \leq s \tag{5}$$

hold.

If $R = r\sqrt{2}$, then the quadrilateral is square, both circles are concentric and inequality hold.

If $R \neq r\sqrt{2}$, then the equality holds if and only if the quadrilateral is an isosceles trapezoid.

The inequality

$$s \leq s_2 \tag{6}$$

hold with equality if and only if the quadrilateral is orthodiagonal.

The inequalities

$$s_1 \leq s \leq s_2 \tag{7}$$

hold.

If $R = r\sqrt{2}$, then both inequalities becomes equalities and in this case the quadrilateral is square.

If $R \neq r\sqrt{2}$, then at least one of the inequality is strict.

Theorem 3 (Fejes Tóth Inequality, see [4] page 147 or [7] page 166)**.** *In a bicentric quadrilateral, the inequality*

$$R \geq \sqrt{2}r \tag{8}$$

hold.

2. Main results and applications

Lemma 1. *In every bicentric quadrilateral the following inequalities*

$$\sqrt{4R^2 + r^2} \leq 2R + \left(3 - 2\sqrt{2}\right) r \tag{9}$$

and

$$\sqrt{4R^2 + r^2} \geq \frac{4\sqrt{2}}{3} R + \frac{1}{3} r \tag{10}$$

hold.

Proof. After squaring we obtain $\left(\sqrt{2} - 1\right)^4 r^2 + \left(12 - 8\sqrt{2}\right) Rr \geq r^2$ or $4r\left(\sqrt{2} - 1\right)^2 \left(R - r\sqrt{2}\right) \geq 0$, which is true.

After squaring we obtain $36R^2 + 9r^2 \geq 32R^2 + r^2 + 8\sqrt{2}\, Rr$ or $4\left(R - r\sqrt{2}\right)^2 \geq 0$, which is true.

Lemma 2. *In every bicentric quadrilateral*

$$\frac{4\sqrt{3}}{3} \frac{R}{r} + \frac{1}{3} \leq \sqrt{4\left(\frac{R}{r}\right)^2 + 1} \leq 2\frac{R}{r} + 3 - 2\sqrt{2} \tag{11}$$

hold.

Proof. Inequalities hold from (9) and (10).

Taking (8) into account we have $\frac{R}{r} \geq \sqrt{2}$, so we will study the variations of the function $f : \left[\sqrt{2}, +\infty\right) \to \mathbb{R}$, $f(x) = \sqrt{4x^2+1}$. We have that

$f'(x) = \frac{4x}{\sqrt{4x^2+1}}$, $f''(x) = \frac{4}{(4x^2+1)\sqrt{4x^2+1}}$ and the following variation table

x	$\sqrt{2}$										
f'			$+$		$+$		$+$		$+$		
f	3	$\cup$	$\nearrow$		$\cup$		$\nearrow$	$\cup$	$\nearrow$		$+\infty$
f''		$+$		$+$		$+$		$+$		$+$	$+$

The function f has the line $d : y = 2x$ oblique asymptote and the following graph representation from Fig.1.
The line d_2 has the equation $d_2 : y = 2x + 3 - 2\sqrt{2}$, $d_2 \| d$ and the line d_1 is tangent to the graph of the function f at the point $A\left(\sqrt{2}, 3\right)$.

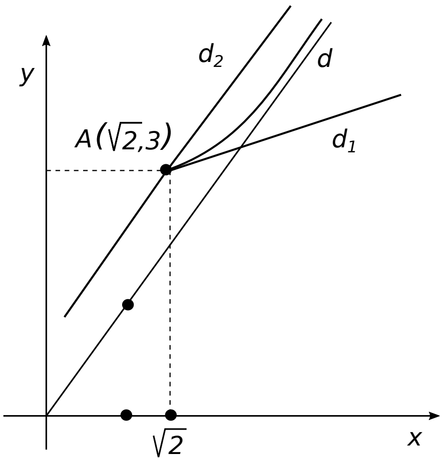

Figure 1

The equation of the line d_1 is $d_1 : y = \frac{4\sqrt{2}}{3}x + \frac{1}{3}$.
The geometric interpretation is that the graph of the function f is between the lines d_1 and d_2, so

$$\frac{4\sqrt{2}}{3}x + \frac{1}{3} \leq \sqrt{4x^2+1} \leq 2x + 3 - 2\sqrt{2} \tag{12}$$

If in (12) instead of x we put $\frac{R}{r}$, we get the inequalities from (11).

Theorem 4. *In every bicentric quadrilateral hold*

$$\begin{aligned}\frac{\left(\sqrt{4R^2+r^2}-r\right)^2}{2R^2r^2} &\leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2} \leq \\ &\leq \frac{\left(2R^2+r^2\right)\sqrt{4R^2+r^2}\, - 4R^2r + r^3}{4R^2r^3}\end{aligned} \tag{13}$$

Proof. By using the identities (1)–(4) and the inequality (7), we have

$$\sum \frac{1}{a^2} = \left(\sum \frac{1}{a}\right)^2 - 2\sum \frac{1}{ab} = \left(\frac{\sum abc}{abcd}\right)^2 - \frac{2\sum ab}{abcd} =$$

$$= \frac{s^2\left(2r^2 + 2r\sqrt{4R^2+r^2}\right)^2}{s^4r^4} - \frac{2\left(s^2 + 2r^2 + 2r\sqrt{4R^2+r^2}\right)}{s^2r^2} =$$

$$= \frac{2}{r^2t}\left(8R^2 + 2r^2 + 2r\sqrt{4R^2+r^2} - t\right),$$

where $t = s^2 \in [s_1^2, s_2^2]$.

Let $f : [s_1^2, s_2^2] \to \mathbb{R}$ be a function defined by
$f(t) = \frac{2}{r^2t}\left(8R^2 + 2r^2 + 2r\sqrt{4R^2+r^2} - t\right)$.
Since f is decreasing we obtain $f(s_2^2) \le f(s^2) \le f(s_1^2)$, $s^2 \in [s_1^2, s_2^2]$.

$$f(s_1^2) = \frac{2}{r^2s_1^2}\left(8R^2 + 2r^2 + 2r\sqrt{4R^2+r^2} - s_1^2\right) =$$

$$= \frac{2\left(8R^2 + 2r^2 + 2r\sqrt{4R^2+r^2} - 8r\sqrt{4R^2+r^2} + 8r^2\right)}{8r^3\left(\sqrt{4R^2+r^2} - r\right)} =$$

$$= \frac{\left(8R^2 + 10r^2 - 6r\sqrt{4R^2+r^2}\right)\left(\sqrt{4R^2+r^2} + r\right)}{4r^3 \cdot 4R^2} =$$

$$= \frac{(2R^2 + r^2)\sqrt{4R^2+r^2} - 4R^2r + r^3}{4R^2r^3}$$

and

$$f(s_2) = \frac{2}{r^2s_2^2}\left(8R^2 + 2r^2 + 2r\sqrt{4R^2+r^2} - s_2^2\right) =$$

$$= \frac{2\left(8R^2 + 2r^2 + 2r\sqrt{4R^2+r^2} - 4R^2 - 2r^2 - 2r\sqrt{4R^2+r^2}\right)}{r^2\left(\sqrt{4R^2+r^2} + r\right)^2} =$$

$$= \frac{8R^2\left(\sqrt{4R^2+r^2} - r\right)^2}{r^2 \cdot 16R^4} = \frac{\left(\sqrt{4R^2+r^2} - r\right)^2}{2R^2r^2},$$

so (13) follows.

Corollary 1. *In every bicentric quadrilateral holds*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \ge \frac{\left(\sqrt{4R^2+r^2} - r\right)^2}{2R^2r^2} \ge \frac{4}{s_1r} \ge \frac{4}{F}, \tag{14}$$

where $s_1 = \sqrt{8r\left(\sqrt{4R^2+r^2} - r\right)}$.

Proof. The first inequality represent the left side of inequality from Theorem 4. If we denote with $x = \frac{R}{r} \ge \sqrt{2}$, the third inequality is equivalent, with $\left(\sqrt{4x^2+1} - 1\right)^5 \ge 8x^4$, which is true if $x \ge \sqrt{2}$ (according Wolphram Alpha). The last inequality represent the left side of inequality (7).

Corollary 2. *In every bicentric quadrilateral holds*

$$(15)\quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\geq\frac{\left(\sqrt{4R^2+r^2}-r\right)^2}{2R^2r^2}\geq\frac{2\left(2\sqrt{2}R-r\right)^2}{9R^2r^2}\geq\frac{4}{s_1r}\geq\frac{4}{F}.$$

Proof. Inequality $\dfrac{\left(\sqrt{4R^2+r^2}-r\right)^2}{2R^2r^2}\geq\dfrac{2\left(2\sqrt{2}R-r\right)^2}{9R^2r^2}$ holds from (10). Inequality $\dfrac{2\left(2\sqrt{2}R-r\right)^2}{9R^2r^2}\geq\dfrac{4}{s_1r}$ is equivalent with $\left(2\sqrt{2}R-r\right)^2 s_1r\geq 18R^2r^2$ or after squaring, $\left(2\sqrt{2}\,x-1\right)^2 8\left(\sqrt{4x^2+1}-1\right)\geq 4\cdot 81x^4$, where $x=\dfrac{R}{r}\geq\sqrt{2}$ or $2\left(2\sqrt{2}x-1\right)^2\left(\sqrt{4x^2+1}-1\right)\geq 81x^4$ which is true, according WA, for any $x\geq\sqrt{2}$.
The other inequalities it results from Corollary 1.

Corollary 3. *In every bicentric quadrilateral the inequality*

$$(16)\quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\leq\frac{2R^3+\left(1-2\sqrt{2}\right)R^2r+Rr^2+\left(2-\sqrt{2}\right)r^3}{2R^2r^3}$$

holds.

Proof. It follows from right side of inequality from Theorem 4 and inequality (9).

We have $\sum a^4=\sigma_1^4-4\sigma_1^2\sigma_2+4\sigma_1\sigma_3+2\sigma_2^2-4\sigma_4$ and replacing the equalities from Theorem 1, we obtain

$$(17)\quad \sum a^4=2\left(s^2+2r^2+2r\sqrt{4R^2+r^2}\right)^2-4s^2\left(5r^2+4r\sqrt{4R^2+r^2}\right).$$

Also $\sum(a-b)^4=3\sigma_1^4-16\sigma_1^2\sigma_2+4\sigma_1\sigma_3+20\sigma_2^2-16\sigma_4$ and replacing, after perform some calculation, we obtain

$$(18)\quad \begin{aligned}\sum(a-b)^4=20\left(s^2+2r^2+2r\sqrt{4R^2+r^2}\right)^2-\\ -16s^2\left(s^2+8r^2+7r\sqrt{4R^2+r^2}\right).\end{aligned}$$

Remark 1. If $\sum(a-b)^4=0$, equivalent with $a=b=c=d$, so $ABCD$ is square. Then $R=2\sqrt{2}$ and it's immediately that $s_1=s_2=4r$ and $\sum a^4-4F^2=0$.

Theorem 5. *The best constant k such that the inequality*

$$(19)\quad \sum a^4\geq 4F^2+k\sum(a-b)^4$$

it's true in every bicentric quadrilateral is $k=\dfrac{1}{2}$.

Proof. Taking Remark 1 into account, it follows that for $R=r\sqrt{2}$ inequality (19) becomes equality. So for $R=r\sqrt{2}$, inequality from (19) is true for any real number k. In the following, we will consider that $R\neq r\sqrt{2}$, so $R > r\sqrt{2}$. In this situation, inequality from statement is equivalent, according (17), (18), $F = sr$ and $t = s^2$ with

$$k \le f(t) = \frac{\left(t + 2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 - 2\left(6r^2 + 4r\sqrt{4R^2 + r^2}\right)t}{10\left(t + 2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 - 8t\left(t + 8r^2 + 7r\sqrt{4R^2 + r^2}\right)},$$

for any $t \in [s_1^2, s_2^2]$, with

$$f'(t) =$$

$$= \frac{-2r\left(\sqrt{4R^2+r^2}+r\right)\left(t^2-\left(8r\sqrt{4R^2+r^2}+8r^2\right)t+48R^2r^2+40r^3\sqrt{4R^2+r^2}+40r^4\right)}{\left(t^2 - 8r\sqrt{4R^2+r^2}t - 12r^2t + 80r^2R^2 + 40r^3\sqrt{4R^2+r^2} + 80r^4\right)^2}.$$

Let $g(t) = t^2 - \left(8r\sqrt{4R^2 + r^2} + 8r^2\right)t + 48R^2r^2 + 40r^3\sqrt{4R^2 + r^2} + 40r^4$ and then we have $t_v = 4r\sqrt{4R^2 + r^2} + 4r^2 < t_1 = 8r\left(\sqrt{4R^2 + r^2} - r^2\right)$, or $4\sqrt{4R^2 + r^2} > 12r$, or $R > \sqrt{2}\,r$, which is true, where t_v is the abscissa of the peak of the function g of degree II in t.

For the function g, the discriminant is $\Delta_g = \left(8r\sqrt{4R^2 + r^2} + 8r^2\right)^2 - 4\left(48R^2r^2 + 40r^3\sqrt{4R^2 + r^2} + 40r^4\right)$ and after calculation $\Delta g = R^2 - 2r^2$. Since $R > r\sqrt{2}$ it results that $\Delta_g > 0$, so the function g has two distinct real roots.

Let $t_1' < t_2'$ the roots of equation $f'(t) = 0$ and $t_1 = s_1^2$, $t_2 = s_2^2$. Then the variation table have the forms

t			t_1'			t_v		t_2'		t_1		t_2
$f'(t)$	$-$	$-$	0	$+$	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$
$f(t)$			$f(t_1')$							$f(t_1)$	$\searrow$	$f(t_2)$

or

t			t_1'			t_v		t_1		t_2		t_2'
$f'(t)$	$-$	$-$	0	$+$	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$
$f(t)$			$f(t_1')$					$f(t_1)$	$\nearrow$	$f(t_2)$		$f(t_2')$

or

t			t_1'			t_v		t_1		t_2'		t_2
$f'(t)$	$-$	$-$	0	$+$	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$
$f(t)$			$f(t_1')$					$f(t_1)$	$\nearrow$	$f(t_2')$	$\searrow$	$f(t_2)$

so $\inf\limits_{t_1 \le t \le t_2} f(t) = \min\{f(t_1), f(t_2)\}$ and $f(t) \ge \min\{f(t_1), f(t_2)\}$, for any $t \in [s_1^2, s_2^2]$.

In the following we denote $\dfrac{R}{r} = x$ and since $R > 2\sqrt{2}$, we have that $x > \sqrt{2}$. We calculate the expression

$$f(t_1) = f(s_1^2) = \frac{\left(s_1^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 - 2\left(6r^2 + 4r\sqrt{4R^2 + r^2}\right)s_1^2}{10\left(s_1^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 - 8s_1^2\left(s_1^2 + 8r^2 + 7r\sqrt{4R^2 + r^2}\right)}$$

with $s_1 = \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}$, we have

$$f(t_1) = \frac{4r^2\left(5\sqrt{4R^2+r^2} - 3r\right)^2 - 32r^2\left(3r + 2\sqrt{4R^2+r^2}\right)\left(\sqrt{4R^2+r^2} - r\right)}{40r^2\left(5\sqrt{4R^2+r^2} - 3r\right)^2 - 960r^2\left(\sqrt{4R^2+r^2} - r\right)\sqrt{4R^2+r^2}} =$$

$$= \frac{\left(5\sqrt{4x^2+1} - 3\right)^2 - 8\left(3 + 2\sqrt{4x^2+1}\right)\left(\sqrt{4x^2+1} - 1\right)}{10\left(\left(5\sqrt{4x^2+1} - 3\right)^2 - 24\left(\sqrt{4x^2+1} - 1\right)\sqrt{4x^2+1}\right)} =$$

$$= \frac{36x^2 + 42 - 38\sqrt{4x^2+1}}{10\left(4x^2 + 10 - 6\sqrt{4x^2+1}\right)} = \frac{\left(9\sqrt{40x^2+1} - 11\right)\left(\sqrt{4x^2+1} - 3\right)}{10\left(\sqrt{4x^2+1} - 3\right)^2},$$

from where $f(t_1) = \dfrac{9\sqrt{4x^2+1} - 11}{10\left(\sqrt{4x^2+1} - 3\right)}$. In the same way we observe that $f(t_2) = \dfrac{\sqrt{4x^2+1} + 1}{2\left(\sqrt{4x^2+1} - 3\right)}$. We make the observation that since $x > \sqrt{2}$ it follows that $\sqrt{4x^2+1} - 3 > 0$.

By calculation $f(t_1) - f(t_2)$ we get

$$f(t_1) - f(t_2) = \frac{4\sqrt{4x^2+1} - 16}{10\left(\sqrt{4x^2+1} - 3\right)} = \frac{2(4x^2 - 15)}{5\left(\sqrt{4x^2+1} - 3\right)\left(\sqrt{4x^2+1} + 4\right)},$$

where $x = \dfrac{R}{r} > \sqrt{2}$.

Let $u : \left(\sqrt{2}, \infty\right) \to \mathbb{R}$ be a function, defined by $u(x) = \inf\limits_{t_1 \le t \le t_2} f(t) = \min\{f(t_1), f(t_2)\}$ and taking above into account we have
$$u(x) = \begin{cases} \dfrac{9\sqrt{4x^2+1} - 11}{10\left(\sqrt{4x^2+1} - 3\right)}, & x \in \left(\sqrt{2}, \dfrac{\sqrt{15}}{2}\right) \\[2ex] \dfrac{\sqrt{4x^2+1} + 1}{2\left(\sqrt{4x^2+1} - 3\right)}, & x \in \left[\dfrac{\sqrt{15}}{2}, \infty\right). \end{cases}$$

It is immediately verified that the function u is strictly increasing on $\left(\sqrt{2}, \infty\right)$ and $\lim\limits_{x \to \infty} u(x) = \dfrac{1}{2}$. From the above it follows that $f(t) > \dfrac{1}{2}$, for any $t \in [s_1^2, s_2^2]$ and $\dfrac{R}{r} \neq \sqrt{2}$. Taking into account the observation from the beginning of the proof, the equality occurs in (19) if $R = r\sqrt{2}$ for any real number k and the conclusion of the theorem results.

In conclusion, equality in (19) occurs if and only if the bicentric quadrilateral becomes a square.

In the following we will give a direct proof of inequality (19) for $k = \dfrac{1}{2}$.

Corollary 4. *In every bicentric quadrilateral the inequality*

$$\sum a^4 \ge 4F^2 + \frac{1}{2}\sum (a - b)^4 \tag{20}$$

holds.

Proof. From (17) and (18), inequality from statement is equivalent with

$$\left(t+2r^2+2r\sqrt{4R^2+r^2}\right)^2-2t\left(6r^2+4r\sqrt{4R^2+r^2}\right)\geq$$
$$\geq 5\left(t+2r^2+2r\sqrt{4R^2+r^2}\right)^2-4t\left(t+8r^2+7r\sqrt{4R^2+r^2}\right),$$

equivalent with

$$4r\left(\sqrt{4R^2+r^2}+r\right)\left[t-4r\left(\sqrt{4R^2+r^2}+r\right)\right]\geq 0. \tag{21}$$

From (5) we have $t=s^2\geq s_1^2$ and we will prove that $s_1^2\geq 4r\left(\sqrt{4R^2+r^2}+r\right)$. The last inequality is equivalent with $8r\left(\sqrt{4R^2+r^2}-r\right)\geq 4r\left(\sqrt{4R^2+r^2}+r\right)$, equivalent with $\sqrt{4R^2+r^2}\geq 3r$ which is true since $R\geq 2r$. From the inequality above, the inequality (21) follows.

Lemma 3. *In every bicentric quadrilateral the identity*

$$(a-b)^4+(b-c)^4+(c-d)^4+(d-a)^4= \tag{22}$$
$$=2s^4-16r\sqrt{4R^2+r^2}\,s^2+4\left(2r^2+2r\sqrt{4R^2+r^2}\right)^2$$

holds.

Proof. We have

$$\sum_{cyclic}(a-b)^4=\sum_{cyclic}(a^4-4a^3b+6a^2b^2-4ab^3+b^4)=$$
$$=2\sum a^4-4(a^3b+ba^3+b^3c+bc^3+c^3d+d^3c+d^3a+a^3d)+$$
$$+6(a^2b^2+b^2c^2+c^2d^2+d^2a^2)$$

and taking $a+c=b+d$ into account

$$a^3b+b^3a+b^3c+bc^3+c^3d+d^3c+d^3a+a^3d=$$
$$=a^3(b+d)+b^3(a+c)+c^3(b+d)+d^3(a+c)=$$
$$=s(a^3+b^3+c^3+d^3)=s\left[(a+c)(a^2+c^2-ac)+(b+d)(b^2+d^2-bd)\right]=$$
$$=s^2\left[a^2+b^2+c^2+d^2-(ac+bd)\right].$$

But $ac+bd=2r\left(\sqrt{4R^2+r^2}+r\right)$ (see [4] page 25, or [7] page 163) and $\sum a^2=2s^2-4r^2-4r\sqrt{4R^2+r^2}$ (see [4] page 57, or [7] page 164). So

$$\sum_{cyclic}(a^3+b^3a)=s^2\left(2s^2-4r^2-4r\sqrt{4R^2+r^2}-2r\sqrt{4R^2+r^2}-2r^2\right)=$$
$$=s^2\left(2s^2-6r\sqrt{4R^2+r^2}-6r^2\right),$$
$$\sum a^4=2\left(s^2+2r^2+2r\sqrt{4R^2+r^2}\right)^2-4s^2\left(5r^2+4r\sqrt{4R^2+r^2}\right)$$

and

$$\sum_{cyclic}a^2b^2=(a^2+c^2)(b^2+d^2)=\left[(a+c)^2-2ac\right]\left[(b+c)^2-2bd\right]=$$
$$=(s^2-2ac)(s^2-2bd)=s^4-2(ac+bd)s^2+4abcd=$$
$$=s^4-\left(4r\sqrt{4R^2+r^2}+4r^2\right)s^2+4r^2s^2.$$

From the above identities, we have

$$\begin{aligned}\sum_{cyclic}(a-b)^4 &= 4\left(s^2+2r^2+2r\sqrt{4R^2+r^2}\right)^2-\left(40r^2+32r\sqrt{4R^2+r^2}\right)s^2-\\ &-8s^4+\left(24r\sqrt{4R^2+r^2}+24r^2\right)s^2+6s^4-\\ &-\left(24r\sqrt{4R^2+r^2}+24r^2\right)s^2+24r^2s^2=\\ &=4s^4+\left(16r^2+16r\sqrt{4R^2+r^2}\right)s^2+4\left(2r^2+2r\sqrt{4R^2+r^2}\right)^2-\\ &-\left(40r^2+32r\sqrt{4R^2+r^2}\right)s^2-8s^4+\left(24r\sqrt{4R^2+r^2}+24r^2\right)s^2+\\ &+6s^4-\left(24r\sqrt{4R^2+r^2}+24r^2\right)s^2+24r^2s^2=\\ &=2s^4+\Big(16r^2+16r\sqrt{4R^2+r^2}-40r^2-32r\sqrt{4R^2+r^2}+\\ &+24r\sqrt{4R^2+r^2}+24r^2-24r\sqrt{4R^2+r^2}-24r^2+24r^2\Big)s^2+\\ &+4\left(2r^2+2r\sqrt{4R^2+r^2}\right)^2,\end{aligned}$$

from where (22) it follows.

Theorem 6. *The best constant k such that the inequality*

$$\sum a^4 \geq 4F^2+k\sum_{cyclic}(a-b)^4 \tag{23}$$

is true in every bicentric quadrilateral is $k=1$.

Proof. It is immediately checked that if $R=r\sqrt{2}$, the quadrilateral is a square, then the inequality (23) becomes equality, for any real number k. In the following, we will consider $R\neq r\sqrt{2}$, so $R>r\sqrt{2}$. In this situation, from the statement, Lemma 3 and (17) we have

$$\frac{2\left(s^2+2r^2+2r\sqrt{4R^2+r^2}\right)^2-4s^2\left(5r^2+4r\sqrt{4R^2+r^2}\right)-4s^2r^2}{2s^4-16r\sqrt{4R^2+r^2}s^2+4\left(2r^2+2r\sqrt{4R^2+r^2}\right)^2}\geq k,$$

for any $s\in[s_1,s_2]$.
We denote $s^2=t\in[s_1^2,s_2^2]$ and consider the function $f:[s_1^2,s_2^2]\to\mathbb{R}$

$$f(t)=\frac{\left(t+2r^2+2r\sqrt{4R^2+r^2}\right)^2-2t\left(6r^2+4r\sqrt{4R^2+r^2}\right)^2}{t^2-8r\sqrt{4R^2+r^2}t+2\left(2r^2+2r\sqrt{4R^2+r^2}\right)^2}$$

and then

$$\begin{aligned}f'(t)=-4r\Big(&\left(\sqrt{4R^2+r^2}-2r\right)t^2-\left(4r^2\sqrt{4R^2+r^2}+8R^2r+4r^3\right)t+\\ &+32r^4\sqrt{4R^2+r^2}+64R^2r^3+32r^5\Big):\\ &:\left(t^2-8r\sqrt{4R^2+r^2}\,t+16r^2\left(r\sqrt{4R^2+r^2}+2R^2+r^2\right)\right)^2.\end{aligned}$$

Let $x = \dfrac{R}{r} > \sqrt{2}$ and the function $g : [s_1^2, s_2^2] \to \mathbb{R}$ defined by

$$g(t) = \left(\sqrt{4R^2 + r^2} - 2r\right) t^2 - \left(4r^2\sqrt{4R^2 + r^2} + 8R^2 r + 4r^3\right) t +$$
$$+ 32r^4\sqrt{4R^2 + r^2} + 64R^2r^3 + 32r^5, \quad t \in [s_1^2, s_2^2].$$

We have to prove that $t_v < t_1$, equivalent with

$$2\sqrt{4x^2 + 1} + 4x^2 + 2 < 8\left(\sqrt{4x^2 + 1} - 1\right)\left(\sqrt{4x^2 + 1} - 2\right), \qquad \text{or}$$

$$\sqrt{4x^2 + 1} + 2x^2 + 1 < 4\left(\sqrt{4x^2 + 1} - 1\right)\left(\sqrt{4x^2 + 1} - 2\right), \qquad \text{or}$$

$13\sqrt{4x^2 + 1} < 14x^2 + 11$, and after squaring $4(x^2 - 2)(49x^2 + 6) > 0$. Since $\sqrt{4R^2 + r^2} - 2r > 0$, it results that g is increasing on $[s_1^2, s_2^2]$, or

$$g(t) \geq g(s_1^2) = 32r^5\left(6x^2 + 3 - 5\sqrt{4x^2 + 1}\right)\left(\sqrt{4x^2 + 1} - 2\right) > 0,$$

for any $x > \sqrt{2}$.
So f is decreasing for any $t \in [s_1^2, s_2^2]$. We have $k \leq f(s_2^2) \leq f(t)$, for any $t \in [s_1^2, s_2^2]$ and the best constant is $k = \inf\limits_{x \in \left(\sqrt{2}, +\infty\right)} h(x) = 1$, where $h : \left(\sqrt{2}, \infty\right) \to \mathbb{R}$ defined by $h(x) = f(s_2^2) = \dfrac{\sqrt{4x^2 + 1} + 1}{\sqrt{4x^2 + 1} - 3}$, $h'(x) < 0$ for any $x \in \left(\sqrt{2}, \infty\right)$ and $\lim\limits_{x \to \infty} h(x) = 1$.
Taking into account the observation from the beginning of the proof, the equality occurs in (23), if $R = r\sqrt{2}$ for any real number k and the conclusion of the theorem results.

Corollary 5. *In every bicentric quadrilateral, the inequality*

$$a^4 + b^4 + c^4 + d^4 \geq 4F + (a - b)^4 + (b - c)^4 + (c - d)^4 + (d - a)^4 \tag{24}$$

holds.

Proof. The inequality (24) is obtained from Theorem 6 for $k = 1$. In the following, we give a direct proof of inequality (23) for $k = 1$. From Lemma 3 and (17), inequality from statement is equivalent with

$$\left(t + 2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 - 2\left(5r^2 + 4r\sqrt{4R^2 + r^2}\right) t \geq$$
$$\geq 2tr^2 + t^2 - 8r\sqrt{4R^2 + r^2}t + 2\left(2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2,$$

equivalent with

$$\left(\sqrt{4R^2 + r^2} - 2r\right) t \geq 4R^2 r + 2r^2\sqrt{4R^2 + r^2} + 2r^3.$$

Since $t \geq s_1^2 = 8r\left(\sqrt{4R^2 + r^2} - r\right)$ it will sufficient to prove that

$$8\left(\sqrt{4x^2 + 1} - 1\right)\left(\sqrt{4x^2 + 1} - 2\right) \geq 4x^2 + 2 + 2\sqrt{4x^2 + 1}, \text{for any } x \geq \sqrt{2}.$$

If we denote $\sqrt{4x^2 + 1} = y \geq 3$, then the above inequality becomes $8(y - 1)(y - 2) \geq y^2 + 1 + 2y$ or $(7y - 5)(y - 3) \geq 0$, which is true since $y \geq 3$.

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